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On the conditions of potentiality in finite elasticity and hypo-elasticity

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Abstract

Thermodynamic approaches to finite elasticity are almost generally accepted. Nevertheless, there is still a lack of proof for the necessity of potential strain-stress relations in generally defined elasticity and hypo-elasticity. This situation has resulted in ambiguous applications of the general concept of elasticity to the description of irreversible phenomena in viscoelastic solids and liquids. This paper makes a brief review of the general concepts of elasticity and hypo-elasticity, with most of the attention paid to the Eulerian description, employed in viscoelastic theories. Then it is demonstrated that all hypothetical materials with non-potential finite elastic or hypo-elastic constitutive relations can create an energy from nothing, i.e. work as perpetual motion machines. This gives a 'physical' proof of necessity of potential conditions in general finite elasticity and hypo-elasticity and their extensions to finite viscoelasticity.

Keywords: Elasticity; Hypo-elasticity; Hyperelasticity; Potentiality

1. Introduction

This paper is motivated by the fact that many theories of irreversible phenomena in continua include as a 'well-known part', the concept of finite elasticity. These are the theories of nonlinear viscoelasticity in solids, e.g. polymer glasses and cross-linked rubbers (Green and Rivlin, 1957; Treloar, 1975; Wineman and Waldron, 1993; Drozdov, 1998), and in liquids, e.g. polymer melts and concentrated solutions (Larson, 1988; Leonov and Prokunin, 1994). In the case of the viscoelastic polymer solids, the concept of elasticity is used for both the 'instantaneous' and equilibrium responses. For viscoelastic liquids, it is employed for the instantaneous elastic response and, for thermodynamically related theories, as a state of 'local thermodynamic equilibrium'. The elasto-viscoplasticity of metals (Naghdi,

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1990; Leonov and Padovan, 1996) represents another type of irreversible theory where the concept of finite elasticity is also important.

A great majority of books on the theory of elasticity (Green and Adkins, 1960; Green and Zerna, 1968; Narasimhan, 1993; Antman, 1995) employ the approach consistent with thermodynamics, where the stress and strain tensors are related through a potential function, i.e. the Helmholtz free energy of deformations. Antman (1995) gives one of the recent clear proofs of the existence of such a potential relation based on the thermomechanics approach (Coleman and Noll, 1963; Coleman, 1964), providing that there is no dissipation in elastic solids in isothermal case.

However, it still does not mathematically necessitate the thermodynamically related approaches. Perhaps this is why in the books by Truesdell and Noll (1992) and Wang and Truesdell (1973) the thermodynamically consistent, potential approach of hyperelasticity coexists with thermodynamically inconsistent, non-potential approach of elasticity and hypo-elasticity. This coexistence has been extended to the case of viscoelastic liquids, where a non-potential single integral constitutive relation proposed by Rivlin and Sawyers (1971), was then 'successfully' tested in several papers (Larson, 1988, section 3.6). More general non-potential approach for viscoelastic liquids had been proposed earlier (Truesdell and Noll, 1992). Evidently, these non-potential approaches are rooted in the general concept of finite elasticity.

The elastic materials are generally defined as simple materials without memory (Truesdell and Noll, 1992). The more narrow definition of elasticity proposed earlier by Novozhilov (1961), includes the important additional condition that the work spent on deformation is independent of a deformation path. This condition was found *sufficient* by Sternberg and Knowles (1979) to prove the existence of elastic potential for a simple material without memory (Novozhilov, 1961).

It seems impossible to find a pure mathematical proof of *necessity* of the potential stress–strain relations within the general definition of elasticity. Therefore, a clear physical proof of the necessity should be given instead. In this regard, the demonstration of pathological, unphysical behavior of thermodynamically inconsistent constitutive models seems to be a proper tool to distinguish the class of potential relations in elasticity as only physically meaningful. This is the main objective of this paper.

The paper is organized as follows. Section 2 gives some preliminaries in continuum mechanics, with more attention paid to Eulerian description useful in the case of rubber elasticity. Section 3 briefly describes the constitutive relations for elastic materials. Here, much attention is paid to the isotropic case employed in rubber elasticity. Section 4 gives a new analysis of hypo-elasticity. Section 5 proves work theorems exposing non-physical features of non-potential approaches. Finally, Section 6 briefly discusses the definition of elastic materials and the stability constraints imposed on hyperelastic constitutive relations.

2. Continuum mechanics preliminary

Consider a material continuum which initially (at time t_0) occupies a domain Ω_{t_0} and at actual time $t\,(t > t_0)$, a domain $\Omega_t\,(\Omega_{t_0}, \Omega_t \subseteq R^3)$. The vector-points $\underline{\xi}$ and $\underline{x}\,(\underline{\xi} \in \Omega_{t_0}, \underline{x} \in \Omega_t)$ mark the 'corresponding material points' belonging to Ω_{t_0} and Ω_t, respectively. We call $\underline{\xi}$ and $\underline{x}$ Lagrangean and Eulerian coordinates of material points in continuum, respectively, and postulate the existence of a local one-to-one mapping,

$$\underline{x} = \underline{x}(t, \underline{\xi}) \quad (t > t_0), \quad \underline{x}(t_0, \underline{\xi}) = \underline{\xi}, \tag{1}$$

except for some special points, lines and surfaces. Eqn (1) which constitutes the *law of motion* for a continuum, contains all the information of its motions.

The relations

$$\delta \underline{x} = \underline{\underline{F}} \cdot \delta \underline{\xi} \quad \text{or} \quad \underline{\underline{F}} = \left(\underline{\nabla}_{\xi} \underline{x}\right)^{\mathrm{T}} \tag{2}$$

define the second rank *strain gradient tensor* $\underline{\underline{F}}$. The subscript T denotes the operation of transposition. The above local one-to-one mapping between $\underline{x}$ and $\underline{\xi}$ means that $\det \underline{\underline{F}} \neq 0$.

The Cayley polar decomposition,

$$\underline{\underline{F}} = \underline{\underline{U}} \cdot \underline{\underline{R}} \tag{3}$$

represents the strain gradient through symmetric, $\underline{\underline{U}}$, and orthogonal $\underline{\underline{R}}$, tensors. The positive definite, symmetric *Finger tensor* $\underline{\underline{B}}$ is then introduced as follows:

$$\underline{\underline{B}} = \underline{\underline{F}} \cdot \underline{\underline{F}}^{\mathrm{T}} = \underline{\underline{U}}^2 \tag{4}$$

It gives the Eulerian representation of the metric tensor in the initial (at time t_0), rest state. The Finger tensor $\underline{\underline{B}}$ is often employed in theories of rubber elasticity (Treloar, 1975).

A *measure of deformation* is defined as any monotone isotropic function of tensor $\underline{\underline{B}}$. It means that the Finger tensor $\underline{\underline{B}}$ itself is a measure of deformation. Other useful Eulerian measures of deformation are:

$$\underline{\underline{C}} = \underline{\underline{B}}^{-1}; \quad \underline{\underline{U}} = \sqrt{\underline{\underline{B}}}; \quad \underline{\underline{H}} = \ln \underline{\underline{U}} = \frac{1}{2} \ln \underline{\underline{B}}; \quad \underline{\underline{G}} = \frac{1}{2}\left(\underline{\underline{\delta}} - \underline{\underline{C}}\right) \tag{5}$$

Here $\underline{\underline{C}}$, $\underline{\underline{U}}$, and $\underline{\underline{H}}$ are the *Green, stretching* and *Hencky* tensors, respectively, and $\underline{\underline{\delta}}$ is the *unit* tensor. In (5) $\underline{\underline{G}}$ is the *Green deformation measure* which is equal to the half of the difference between the fundamental tensors at actual, t, and initial, t_0, time instants. It should be noted that in Eulerian approach, all the vector and tensor fields can be treated without loss of generality as Cartesian.

The Cayley-Hamilton identity,

$$\underline{\underline{B}}^3 - I_1 \underline{\underline{B}}^2 + I_2 \underline{\underline{B}} - I_3 \underline{\underline{\delta}} = \underline{\underline{0}}, \tag{6}$$

introduces the basic invariants, I_k, as follows:

$$I_1 = tr\underline{\underline{B}}; \quad I_2 = 1/2\left(I_1^2 - tr\underline{\underline{B}}^2\right); \quad I_3 = \det \underline{\underline{B}}. \tag{7}$$

Using the local mass conservation, yields:

$$\det \underline{\underline{F}} = \rho_0/\rho, \, I_3 = \left(\rho_0/\rho\right)^2, \tag{8}$$

where ρ and ρ_0 are the respective densities in the actual and initial states of deformation. Due to the first formula in (8), $\det \underline{\underline{F}} > 0$, meaning that the tensor $\underline{\underline{U}}$ is strictly positively definite.

The common definition of velocity in continuum is:

$$\underline{v} = \mathrm{d}\underline{x}/\mathrm{d}t \equiv \partial \underline{x}/\partial t|_{\xi}. \tag{9}$$

Consequently using formulate (9) and (2) yields:

$$\delta \underline{v} = \dot{\boldsymbol{F}} \cdot \delta \underline{\xi} = \dot{\boldsymbol{F}} \cdot \underline{\underline{F}}^{-1} \cdot \delta \underline{x}.$$

Hence the Eulerian definition of the *velocity gradient tensor*, $\underline{\nabla}\mathbf{v}$, is:

$$(\underline{\nabla}\mathbf{v})^{\mathrm{T}} = \dot{\boldsymbol{F}} \cdot \underline{\underline{F}}^{-1}, \quad \text{or} \quad \underline{\nabla}\mathbf{v} = \left(\underline{\underline{F}}^{\mathrm{T}}\right)^{-1} \dot{\boldsymbol{F}}^{\mathrm{T}}. \tag{10}$$

Here the overdot denotes the operation $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \underline{v} \cdot \underline{\nabla}$.

The *evolution equations* for the tensors $\underline{\underline{B}}$ and $\underline{\underline{C}}$ immediately result from (10) as:

$$\overset{\nabla}{\underline{\underline{B}}} \equiv \dot{\underline{\underline{B}}} - \underline{\underline{B}} \cdot \underline{\nabla}\mathbf{v} - (\underline{\nabla v})^{\mathrm{T}} \cdot \underline{\underline{B}} \equiv \overset{0}{\underline{\underline{B}}} - \underline{\underline{B}} \cdot \underline{\underline{D}} - \underline{\underline{D}} \cdot \underline{\underline{B}} = \underline{\underline{0}};$$

$$\left(\overset{0}{\underline{\underline{B}}} \equiv \underline{\underline{B}} \cdot \underline{\underline{\Omega}} - \underline{\underline{\Omega}} \cdot \underline{\underline{B}}\right)$$

$$\overset{0}{\underline{\underline{C}}} \equiv \dot{\underline{\underline{C}}} + \underline{\nabla} v \cdot \underline{\underline{C}} + \underline{\underline{C}} \cdot (\underline{\nabla}\mathbf{v})^{\mathrm{T}} \equiv \overset{0}{\underline{\underline{C}}} + \underline{\underline{D}} \cdot \underline{\underline{C}} + \underline{\underline{C}} \cdot \underline{\underline{D}} = \underline{\underline{0}}. \tag{11}$$

Here the upper symbols, ∇, Δ, and 0 denote the operations of the *upper and lower convected*, and *co-rotational tensor time derivatives*, respectively. The *strain rate*, $\underline{\underline{D}}$, and *vorticity (or spin)*, $\underline{\underline{\Omega}}$, tensors in (11) are defined as:

$$\nabla\mathbf{v} = \underline{\underline{D}} + \underline{\underline{\Omega}}, \quad \underline{\underline{D}} = 1/2\left[\underline{\nabla v} + (\underline{\nabla v})^{\mathrm{T}}\right], \quad \underline{\underline{\Omega}} = 1/2\left[\underline{\nabla v} - (\underline{\nabla v})^{\mathrm{T}}\right]. \tag{12}$$

Eqns (11) contain all the information about the evolution of deformation in continuum, including the law of mass conservation:

$$\partial\rho/\partial t + \underline{\nabla} \cdot (\rho\underline{v}) = 0. \tag{13}$$

This equation can immediately be derived from eqn (11).

In addition to kinematics, the dynamic effects described by surface forces, can be characterized in Eulerian approach by the symmetric Cauchy stress tensor $\underline{\underline{\sigma}}$.

Along with the Eulerian presentation of kinematic tensors and stress tensor, it is also useful to employ their Lagrangean presentation. In this regard, along with basis (covariant) Eulerian vectors $\underline{e}_i(\underline{x})$, one can introduce the Lagrangian basis vectors, $\hat{\mathbf{e}}_i(\underline{\xi}, t)$, which are 'imbedded' in the continuum and travel with it (Sedov, 1965). Denoting the Lagrangian components of tensors by overcaps, the relations between these and Eulerian tensor components are defined as:

$$\underline{\underline{C}} = C_{ij}\underline{e}^i\underline{e}^j = g_{ij}^0\hat{\mathbf{e}}^i\hat{\mathbf{e}}^j; \quad \underline{\underline{B}} = B^{ij}\underline{e}_i\underline{e}_j = g_0^{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j;$$

$$\underline{\underline{\sigma}} = \sigma^{ij}\underline{e}_i\underline{e}_j = \hat{\sigma}^{ij}\hat{\mathbf{e}}_i\hat{\mathbf{e}}_j.$$

$$\underline{\underline{G}} = \gamma_{ij}\underline{e}^i\underline{e}^j = \hat{\gamma}_{ij}\hat{\mathbf{e}}^i\hat{\mathbf{e}}^j; \quad D = D_{ij}\underline{e}^i\underline{e}^j = \hat{D}_{ij}\hat{\mathbf{e}}^i\hat{\mathbf{e}}^j; \tag{14}$$

Here g_{ij}^0 and g_0^{ij} are components of the metric tensor in initial (non-deformed) state at $t = t_0$. It should be mentioned that the Lagrangean representations of the Eulerian tensors shown in eqns (14) are valid only for shown structure of indices (Sedov, 1965). This is because the operations of raising and lowering indices for Eulerian tensors are carried out by the metric tensor $g_{ik} = (\underline{e}_i, \underline{e}_k)$, and for Lagrangean tensors, by $\hat{g}_{ik} = (\hat{\mathbf{e}}_i, \hat{\mathbf{e}}_k)$.

One important kinematic relation (Oldroyd, 1950) readily established for Lagrangean formulation, is:

$$\partial\hat{\gamma}_{ij}/\partial t|_{\xi} = \hat{D}_{ij} \tag{15}$$

Using eqns (14) and (15) the well-known expression for the *rate and variation of local work W* (per mass unit) under action of surface forces can dually be represented as:

$$\rho \dot{W} = tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}\right) = \hat{\sigma}^{ij}\hat{D}_{ij}; \quad \text{or} \quad \rho \, \mathrm{d}W = \hat{\sigma}^{ij} \, \mathrm{d}\hat{\gamma}_{ij} \tag{16}$$

All the above formulae are independent of a choice of constitutive relations. In the following two sections, we specify the type of materials under discussion.

3. Constitutive relations for elastic materials

According to the general definition (Truesdell and Noll, 1992), “a material is called *elastic* if it is simple and if the stress at time t depends only on the local configuration at time t, and not on the entire past history of motion”. The concept of *simple materials*, has been extensively discussed by Truesdell and Noll (1992).

The above general definition of elastic materials yields a generally *anisotropic* constitutive relation between the stress tensor $\underline{\underline{\sigma}}$ and a measure of deformation, which can be written as:

$$\underline{\underline{\sigma}} = \underline{\underline{h}}\left(\underline{\underline{B}}\right) = \underline{\underline{l}}\left(\underline{\underline{G}}\right) = \hat{\underline{\underline{l}}}\left(\hat{\gamma}_{ij}; g_{ij}^{0}\right) \tag{17}$$

In *isotropic case*, eqn (17) is specified so that the stress is an isotropic tensor function of a deformation measure. This can generally be represented in Eulerian approach as:

$$\underline{\underline{\sigma}}\left(\underline{\underline{B}}\right) = 2\underline{\underline{B}} \cdot \sum_{k=1}^{3} \varphi_k \partial I_k / \partial \underline{\underline{B}}; \quad \varphi_k = \varphi_k(I_1, I_2, I_3) \quad (k = 1, 2, 3) \tag{18}$$

Also, due to eqns (11) and (18), eqns (16) can be rewritten in isotropic case in the forms:

$$\rho \dot{W} = tr\left(\underline{\underline{\sigma}} \cdot \dot{\underline{\underline{H}}}\right) = \frac{1}{2} tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{B}}^{-1} \cdot \dot{\underline{\underline{B}}}\right) = \sum_{k=1}^{3} \varphi_k \dot{I}_k;$$

$$\rho \, \mathrm{d}W = tr\left(\underline{\underline{\sigma}} \cdot \mathrm{d}\underline{\underline{H}}\right) = \frac{1}{2} tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{B}}^{-1} \cdot \mathrm{d}\underline{\underline{B}}\right) = \sum_{k=1}^{3} \varphi_k \, \mathrm{d}I_k \tag{19}$$

On the right-hand side of the second formula in (19) presents so called Pfaff's differential form which generally is non-integrable. The Pfaff's forms play important roles in formal thermodynamics, as compared to integrable differential forms related to thermodynamic potentials (Sommerfeld, 1956).

In the particular case of *incompressible elasticity*, when $I_3 = 1$, eqns (17) and (18) should include the additional term, $-p\underline{\underline{\delta}}$, where p is the *isotropic pressure* serving as a Lagrange multiplier to relieve the incompressibility constraint $I_3 = 1$.

Finally, when the stress–strain relations are *potential*, a strain energy function F (per mass unit) exists, such that

$$\rho \dot{W} = tr\left(\underline{\underline{D}} \cdot \underline{\underline{\sigma}}\right) = \rho \dot{F}, \quad \text{or} \quad \rho \, \mathrm{d}W = \hat{\sigma}^{ij} \, \mathrm{d}\hat{\gamma}_{ij} = \rho \, \mathrm{d}F, \tag{20a}$$

$$\rho \, \mathrm{d}W = \underline{\underline{\sigma}} \cdot \mathrm{d}\underline{\underline{H}} = \rho \, \mathrm{d}F. \tag{20b}$$

Eqns (20) result in the stress–strain relations:

$$\hat{\sigma}^{ij} = \frac{\rho}{2}\left(\frac{\partial F}{\partial \hat{\gamma}_{ij}} + \frac{\partial F}{\partial \hat{\gamma}_{ji}}\right), \tag{21a}$$

$$\underline{\underline{\sigma}} = \rho \partial F / \partial \underline{\underline{H}} = 2\rho \underline{\underline{B}} \cdot \partial F / \partial \underline{\underline{B}}, \quad \varphi_k = \partial F / \partial I_k. \tag{21b}$$

In formulae (20) and (21), the isotropic case is specified in eqns (20b) and (21b), with the constitutive relation (21b) obtained by Murnaghan (1937).

Eqns (20) and (21) are in fact the definition of the *hyperelastic solids*. The local strain potential F has the thermodynamic sense of the *Helmholtz free energy deformation function per mass unit*. Eqn (21b) shows that for isotropic elastic solids, the 'true' thermodynamically conjugated variables are the thermodynamic stress, $\underline{\underline{\sigma}}/\rho$ and Hencky strain tensor, $\underline{\underline{H}}$. However, the straightforward use of this tensor in the evolution eqn (11) is rather awkward (Gurtin and Spear, 1983).

Eqns (20) demonstrate that for hyperelastic materials, the work produced by surface forces results in the accumulation of free energy of deformation. Therefore under any isothermal regime of deformation, the hyperelastic solids are non-dissipative.

4. Elastic constitutive relation of rate type: hypo-elasticity

The definition of *hypo-elastic constitutive relation* (Truesdell and Noll, 1992) is:

$$\begin{aligned}\overset{0}{\underline{\underline{\sigma}}} = &\left[a_1 tr\underline{\underline{D}} + a_2 tr(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}) + a_3 tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right)\right]\underline{\underline{\delta}} + \left[a_4 tr\underline{\underline{D}} + a_5 tr(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}) + a_6 tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right)\right]\underline{\underline{\sigma}} \\ &+ \left[a_7 tr\underline{\underline{D}} + a_8 tr(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}) + a_9 tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right)\right]\underline{\underline{\sigma}}^2 + a_{10}\underline{\underline{D}} + a_{11}(\underline{\underline{D}} \cdot \underline{\underline{\sigma}} + \underline{\underline{\sigma}} \cdot \underline{\underline{D}}) + a_{12}\left(\underline{\underline{D}} \cdot \underline{\underline{\sigma}}^2 + \underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right).\end{aligned} \tag{22}$$

Here $\overset{0}{\underline{\underline{\sigma}}}$ is the co-rotational time derivative of stress tensor $\underline{\underline{\sigma}}$. Eqn (22) is a rate type of constitutive relation between the stress $\underline{\underline{\sigma}}$ and the strain rate $\underline{\underline{D}}$ tensors for isotropic solids. The right-hand side in eqn (22) is presented as an isotropic tensor function of $\underline{\underline{\sigma}}$ and $\underline{\underline{D}}$, linear in $\underline{\underline{D}}$, with the scalar coefficients a_k depending on three invariants of stress tensor $\underline{\underline{\sigma}}$:

$$I_1^\sigma = tr\underline{\underline{\sigma}}, \quad I_2^\sigma = (1/2)tr\underline{\underline{\sigma}}^2, \quad I_3^\sigma = (1/3)tr\underline{\underline{\sigma}}^3 \tag{23}$$

Due to the term with the scalar multiplier a_{11} in eqn (22), the stress rate, without loss of generality, can also be written in the form of either upper or lower convected tensor time derivatives.

The incompressible case corresponds to the particular relations in eqn (22):

$$tr\underline{\underline{D}} = 0, \quad \underline{\underline{\sigma}} = -p\underline{\underline{\delta}} + \underline{\underline{\sigma}}_e, \quad a_1 = a_4 = a_7 = 0, \quad \overset{0}{\underline{\underline{\sigma}}} \to \underline{\underline{\sigma}}_e \tag{24}$$

Here p is the isotropic pressure and $\underline{\underline{\sigma}}_e$ is the *extra stress tensor*. In this case, the invariants of stress tensor defined by eqn (23) should also be changed for the respective three invariants of the extra stress tensor $\underline{\underline{\sigma}}_e$. It should be mentioned that in the general nonlinear case, $I_{1_e}^\sigma \equiv tr\underline{\underline{\sigma}}_e \neq 0$. Also, the first formula in (16) for rate of work is still valid here since it is independent of a constitutive relation.

Bernstein (1960) discussed the hypo-elasticity and its relation to the finite elasticity and hyperelasticity (Truesdell and Noll, 1992, sections 99–101). Since the constitutive eqn (22) is isotropic, a simpler procedure as compared to that employed by Bernstein (1960), is proposed below to establish the conditions of potentiality. We will operate below with some differential forms for the rates $\dot{I}_k^\sigma$ defined in eqn (23). Therefore, the conditions of potentiality will be treated as those for integrability.

To obtain the differential forms we perform consequently the three operations: (i) making trace of eqn (22); (ii) making scalar multiplication of eqn (22) by the stress tensor, $\underline{\underline{\sigma}}$; and (iii) making scalar multiplication of eqn (23) by square of the stress tensor, $\underline{\underline{\sigma}}^2$. It is useful to note that due to eqn (23), the following equalities hold true:

$$tr\left(\underline{\underline{\sigma}}^m \cdot \overset{0}{\underline{\underline{\sigma}}}\right) = tr\left(\underline{\underline{\sigma}}^m \cdot \dot{\boldsymbol{\sigma}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{tr\left(\underline{\underline{\sigma}}^{m+1}\right)}{m+1}\right) = \dot{I}^{\sigma}_{m+1} \cdot (m = 0, 1, 2) \tag{25}$$

When performing the operations in (ii) and (iii), the Caley–Hamilton identity is used to express the stress polynomials of order more than two through the second order polynomials, and also represent the basic stress invariants in the Caley–Hamilton identity through the three stress invariants I^{σ}_k in eqn (23). In doing so, eqn (22) yields:

$$\dot{I}^{\sigma}_1 = A_{11} tr\underline{\underline{D}} + A_{12} tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}\right) + A_{13} tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right)$$

$$\dot{I}^{\sigma}_2 = A_{21} tr\underline{\underline{D}} + A_{22} tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}\right) + A_{23} tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right)$$

$$\dot{I}^{\sigma}_3 = A_{31} tr\underline{\underline{D}} + A_{32} tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}\right) + A_{33} tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right) \tag{26}$$

Here A_{ik} are some linear functions with respect to the scalars a_k in eqn (22), depending on the three stress invariants I^{σ}_k defined in eqn (23). Also, due to eqns (13) and (16),

$$tr\underline{\underline{D}} = -\dot{\rho}/\rho, \quad \text{and} \quad tr\left(\underline{\underline{\sigma}} \cdot \underline{\underline{D}}\right) = \rho \dot{W}. \tag{27}$$

Eqns (26) can be treated as a set of inhomogeneous linear algebraic equations with respect to the three quantities: $tr\underline{\underline{D}}$, $tr(\underline{\underline{\sigma}} \cdot \underline{\underline{D}})$, and $tr(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}})$.

If $\det\|A_{ik}\| \neq 0$, these quantities are uniquely expressed as a linear form of the left-hand side of eqn (26):

$$\dot{\rho}/\rho = r_k \dot{I}^{\sigma}_k, \quad \dot{W} = (w_k/\rho)\dot{I}^{\sigma}_k, \quad tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right) = s_k \dot{I}^{\sigma}_k. \tag{28}$$

Here the coefficients r_k, w_k and s_k are known functions of the stress invariants I^{σ}_k defined in eqn (24), as soon the constitutive scalars a_k in eqn (23) are known.

Considering now only the first and second equations in (28), one can readily establish the conditions of potentiality/integrability:

$$\frac{\partial r_k}{\partial I^{\sigma}_j} = \frac{\partial r_j}{\partial I^{\sigma}_k}, \quad \frac{\partial}{\partial I^{\sigma}_j}\left(\frac{w_k}{\rho}\right) = \frac{\partial}{\partial I^{\sigma}_k}\left(\frac{w_j}{\rho}\right) \cdot \quad (k, j = 1, 2, 3) \tag{29}$$

Indeed, if the first conditions of potentiality in eqns (29) holds, then the first differential form there is integrable and therefore the density depends only on the stress invariants I^{σ}_k in eqn (25). It means that a function $\rho = \rho(I^{\sigma}_1, I^{\sigma}_2, I^{\sigma}_3)$ does exist. Then substituting it into the second conditions of potentiality in eqn (29) ensures the existence of the *potential* function $\Psi(I^{\sigma}_1, I^{\sigma}_2, I^{\sigma}_3)$. Evidently, both conditions in eqns (29) are necessary and sufficient for the potentiality.

When $\det\|A_{ik}\| = 0$, eqns (28) can also in principle, be obtained when using higher powers of multipliers $\underline{\underline{\sigma}}^n (n > 2, 3)$ in the above operations (i) and (ii). However, it will only show that the hypo-elastic constitutive relation (22) is not *robust*.

The same procedure, but using only steps (i) and (ii) is applicable under conditions (24) to the incompressible hypo-elasticity. In this case, only two equations of type (26) hold:

$$\dot{I}_1^\sigma = B_{11}\rho_0\dot{W} + B_{12}tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right), \quad \dot{I}_2^\sigma = B_{21}\rho_0\dot{W} + B_{22}tr\left(\underline{\underline{\sigma}}^2 \cdot \underline{\underline{D}}\right) \tag{30}$$

Here the coefficients B_{ik} have the explicit expressions through a_k in eqn (23):

$$B_{11} = 3a_1 + I_1^\sigma a_3 + I_2^\sigma a_5 + a_8, \quad B_{12} = 3a_2 + I_1^\sigma a_4 + I_2^\sigma a_6 + a_9,$$

$$B_{21} = I_1^\sigma a_1 + I_2^\sigma a_3 + I_3^\sigma a_5 + \left[I_2^\sigma - \left(I_1^\sigma\right)^2\right]a_9/2 + a_{10},$$

$$B_{22} = I_1^\sigma(a_2 + a_9) + I_2^\sigma a_4 + I_3^\sigma a_6 + a_8 \tag{31}$$

We assume that for the linear set of eqns (30),

$$\Delta \equiv \det\|B_{ik}\| = B_{11}B_{22} - B_{12}B_{21} \neq 0.$$

Then the solution of eqn (30) for rate of work is:

$$\rho_0\dot{W} = \left(B_{22}\dot{I}_1^\sigma - B_{12}\dot{I}_2^\sigma/2\right)/\Delta$$

Thus the condition of integrability (potentiality) is:

$$\frac{\partial}{\partial I_1^\sigma}\left(\frac{B_{12}}{\Delta}\right) + 2\frac{\partial}{\partial I_2^\sigma}\left(\frac{B_{22}}{\Delta}\right) = 0 \tag{32}$$

It means that in the incompressible case, the *potential* function $\Psi(I_1^\sigma, I_2^\sigma, I_3^\sigma)$ also exists.

The above results show that the potential hypo-elastic constitutive relations (22), (23) or (22)–(24) are the constitutive relations for hyperelasticity. This immediately follows from the potentiality:

$$\mathrm{d}W = tr\left[\left(\underline{\underline{\sigma}}/\rho\right) \cdot \mathrm{d}\underline{\underline{H}}\right] = \mathrm{d}\Psi(\underline{\underline{\sigma}}). \tag{33}$$

Eqn (33) displaying the existence of (generally multi-valued) function $\underline{\underline{H}}(\underline{\underline{\sigma}})$, shows that the potential case of hypo-elasticity is a type of hyperelasticity, with

$$\Psi(\underline{\underline{\sigma}}) = F\left(\underline{\underline{B}}(\underline{\underline{\sigma}})\right). \tag{34}$$

5. Work theorems

As mentioned, Knowles and Sternberg (1977) have proved that an elastic material is a hyperelastic under additional (sufficient) condition that the work spent on any cyclic quasi-static deformation is equal to zero. We now make one step forward to demonstrate, complimentary to Knowles and Sternberg (1977), that non-potential approaches in both elasticity and hypo-elasticity are physically meaningless.

5.1. Isotropic elasticity

Consider in the 3D domain $\{I_k > 0\}$ of strain invariants I_k a closed, piece-wise smooth curve Γ. Any such a curve that goes through the rest state R: $\{I_1 = I_2 = 3, I_3 = 1\}$, forms a *closed deformation path*. Let $S(\Gamma)$ be the set of surfaces which can be pulled on the curve Γ. A closed deformation path is called *non-trivial*, if $\inf\{\text{mesS}(\Gamma)\} \neq 0$. It means that a non-trivial closed deformation path forms a loop in the strain invariant domain, which in trivial case degenerates in a simple curve.

The closed deformation path is always trivial when loading and unloading is carried out using only a single deformation mode, such as simple shear, simple elongation, etc. To create a non-trivial deformation path in testing experiments, one should employ a combination of single deformation modes, e.g. to use initially equi-biaxial extension during loading, which is then changed to a uniaxial extension with following unloading in uniaxial extension mode.

5.1.1. Theorem

Any hypothetical isotropic elastic material with general non-potential constitutive relation (18), can produce energy from nothing, i.e. serve as *perpetual motion machine*.

5.1.2. Proof

For any homogeneous quasi-static deformation, the local work ΔW along a nontrivial closed deformation path with a loop-wise contour Γ, due to the second eqn (19) is:

$$\Delta W = \oint_\Gamma (\varphi_k/\rho)\,\mathrm{d}I_k. \tag{35}$$

Let us fix the chosen contour Γ. Since the constitutive eqn (18) is assumed to be non-potential, the integrand in contour integral (35) represents the non-integrable Pfaff's form (19). Then depending on the direction of integration around the contour, the integral (35) is either positive or negative. Let us choose such a direction of integration, starting from the rest state R, that $\Delta W > 0$. This inequality proves the theorem.

5.1.3. Remark

If the integration direction in (35) is chosen so that $\Delta W < 0$, it leads to unexpected 'perpetual' dissipation.

5.1.4. Remark

Due to relation (35), for any hyperelastic material with conditions of potentiality (21b), $\Delta W = 0$.

5.1.5. Remark

The first expression in (19) for the rate of work can also be used, since in this case,

$$\Delta W = \int_{t_1}^{t_2} (\varphi_k/\rho)\,\mathrm{d}\dot{I}_k = \oint_\Gamma (\varphi_k/\rho)\,\mathrm{d}I_k. \tag{36}$$

Here t_1 and t_2 are the time instants of starting and ending the deformation process which forms the closed non-trivial deformation path Γ in (35).

5.1.6. Remark

In the compressible case, closed non-trivial deformation path with the contour Γ, is now defined as a plain curve in the wedge-type 2D domain of invariants I_1 and I_2, with the origin in the point R: $\{I_1 =$

$I_2 = 3\}$ (Green and Adkins, 1960). Then the proof of the theorem 1 is repeated using the modifications of relations (18) with $I_3 = 1$ and $k = 1, 2$.

5.1.7. Remark

On any trivial deformation paths, non-potential finite elasticity and hyperelasticity are not distinguishable, since on these paths, $\Delta W \equiv 0$. The loading and unloading along trivial deformation paths are widely used for testing materials in one type of deformation, such as simple extension, simple shear etc.

The non-potential constitutive relations which resulted in the behavior exposed in the theorem 1, are physically meaningless. Therefore to avoid these unphysical situations, we define *the physically meaningful (or thermodynamically consistent)* elastic constitutive relations as those that produce no work in any quasistatic deformations along any closed nontrivial deformation path (Sternberg and Knowles, 1979).

5.1.8. Theorem

The general isotropic elastic constitutive relations (18) are physically meaningful if, and only if, they are hyperelastic (or potential).

5.1.9. Proof

The *necessity* immediately follows from eqn (35), when the homogeneous quasi-static deformations are chosen for consideration.

The *sufficiency* is evident due to the above definition, since the total work W_{tot} in an elastic body along *all* non-trivial deformation paths confined in its actual volume $V(t)$ $(= \text{mes}\Omega_t)$ is:

$$W_{\text{tot}} = \int_{V(t)} \mathrm{d}\underline{x} \oint_{\Gamma(x)} (\varphi_k/\rho)\, \mathrm{d}I_k = 0. \tag{37}$$

The extension of theorem 2 to the formulation of the local rate of work, $\rho \dot{W}$, follows the relation (36), and to the incompressible case, it is made as in the remark 4 to the theorem 1.

5.2. General (non-isotropic) elasticity

In this case, it is convenient to use the general Langrangean formulae (16) and (17), and define a closed non-trivial deformation path $\hat{\Gamma}_\gamma$ in the entire 6D 'space' $\{-\infty < \hat{\gamma}_{ij} < \infty\}$. The above theorems 1 and 2, along with remarks 1–5, are also hold, since due to (16) and (17),

$$\Delta W = \int_{t_1}^{t_2} \left(\hat{\sigma}^{ij}/\rho\right) \hat{D}_{ij}\, \mathrm{d}t = \oint_{\hat{\Gamma}_\gamma} \left(\hat{\sigma}^{ij}/\rho\right) \mathrm{d}\hat{\gamma}_{ij}, \tag{38}$$

and in the case of non-potentiality, the integrand of the second integral in eqn (38) is a non-integral Pfaff's form.

5.3. Hypo-elasticity

The constitutive relations are given here by eqn (22). First, we make exactly the same definition as above for the *closed non-trivial stress-invariant path* Γ_σ in the 3D 'space' $\{-\infty < I_k^\sigma < +\infty\}$ of the three stress invariants I_k^σ defined in eqn (23). We can then formulate and prove two theorems, similar to the above theorems 1 and 2, since due to the second formula in eqns (29),

$$\Delta W = \int_{t_1}^{t_2} (w_k/\rho)\, \mathrm{d}\dot{I}_k^{\sigma} = \oint_{\Gamma_0} (w_k/\rho)\, \mathrm{d}I_k^{\sigma}, \tag{39}$$

and in the non-potential case, the integrand of the second integral in eqn (39) is a non-integrable Pfaff's form.

We now formulate all the results obtained in this Section, as the general theorem.

5.3.1. Theorem

Constitutive relations for general elasticity or hypo-elasticity are physically meaningful if, and only if, they are hyperelastic (or potential).

6. Concluding remarks

(1) The demonstrations given in Section 5 clearly show that non-potential approaches should be treated as mathematical abstractions having nothing in common with the behavior of real materials. It is evident that the general definition of elasticity/hyperelasticity given by Truesdell and Noll (1992) is not sufficient to single out the only physically meaningful potential approach. Therefore, as the consequence of the theorem 3, the following definition for elasticity is suggested (compare it with that proposed by Novozhilov, 1961): A material is called *elastic* if (i) it is simple; (ii) the stress at time t depends only on the local configuration at time t, and not on the entire past history of motion; and (iii) the work of surface forces spent on a deformation is independent of a deformation path.

(2) The constitutive equations should also obey *the stability constraints*. They have been completely analyzed only for isotropic hyperelastic cases.

The *thermodynamic stability* criteria called 'GNC$^+$ conditions' were established long ago (Truesdell and Noll, 1992, Section 52). Their physical sense is the convexity of the elastic potential with respect to the Hencky strain measure. This condition forbids the non-monotony in the general potential relation $\underline{\underline{H}}(\underline{\underline{\sigma}})$ established after eqn (33).

More general *Hadamard stability* criteria of field equations correspond to the conditions of *strong ellipticity* which coincide with the stability requirements known for dynamic problems. These constraints have been established in both the compressible (Knowles and Sternberg, 1977) and incompressible (Zee and Sternberg, 1983) cases and presented in the form of 'exact' inequalities imposed on the first and second derivatives of elastic potential with respect to strain invariants, I_k. The GCN$^+$ conditions closely associated with the conditions of strong ellipticity, can then be treated as necessary conditions for the strong ellipticity or the Hadamard stability.

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